Optimal Analysis and Design of Structures: Deterministic and Probabilistic Problems

Habilitation booklet

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**Introduction**

Over the years two approaches have been improved to evaluate the uncertainties in engineering structures. The first method is the deterministic design, in which a global factor of safety or a load factor is applied. The second method is the reliability-based design, when the information in respect to the design is known to be within certain limits and have recognized distributions of probability. Although the deterministic design has been favorably used for decades, the appropriate safety is unclear for a given factor of safety. In the reliability-based design, the uncertainties are defined by randomly distributed variables, in which the contribution of occurrence of each feasible value of the variable is examined and the most frequent values of a random variable are related with the highest amounts in the probability density function.

By the use of elastic-plastic analysis and design approaches, considerable saving in material can be achieved. As a result of this advantage, however excessive residual displacements and large plastic deformations could develop, which might lead to failure of the structure. Over the years various limit theorems for the residual displacements and plastic deformations have been recommended in the literature. In this research a proper computational method presented when the complementary strain energy of the residual forces defined as a general measure of the plastic performance of the structures and the residual deformations need to be constrained by considering a limit for the amount of this energy (Kaliszky and Lógó (1997), Lógó et al. (2011) and Movahedi and Lógó (2011)).

In the last seven years after finishing my PhD I have been working mainly in the field of :

- residual plastic limit theorem,
- reliability-based residual plastic limit theorem,
- plastic limit analysis and design of laterally loaded pile structures,
- shakedown analysis and design of laterally loaded pile structures.

1. Residual plastic limit theorem

Let us suppose that the structure has been defined by the concept of plastic analysis and design methods. Accordingly by applying load $P_0$ internal plastic forces $Q^p$ will appear in the structure. When the load is decreased under unloading elastic deformations occur and then the elastic internal forces $-Q^e$ will take place in the structure. Therefore after complete unloading the residual forces will remain in the structure.

$$Q^r = Q^p - Q^e \quad (1)$$

where

$$Q^e = F^{-1} G K^{-1} P_0 \quad (2)$$

here $F$ is the flexibility matrix; $G$ denotes the geometrical matrix; $K$ is the stiffness matrix. Let us suppose the positive-definite function, the complementary energy can be determined from the residual forces.

$$C_r = \frac{1}{2} \sum_{i=1}^{n} \frac{1}{S_i} \int_0^{l_i} (Q_i^p(s) - Q_i^e(s))^2 \, ds \geq 0 \quad (3)$$

here $Q_i^p(s)$ and $Q_i^e(s)$ are the functions of elastic and plastic internal forces; $S_i$ expresses tensile stiffnesses and flexural stiffnesses for trusses and beam members respectively.

A proper computational method proposed that the complementary energy of the residual forces could be defined as a general measure of the plastic performance of the structures and the residual deformations need to be constrained by considering a limit for the amount of this energy:
where $C_{r0}$ is a proper permissible energy value for $C_r$. Now let us consider the case of beam elements

$$ C_r = \frac{1}{2E} \sum_{i=1}^{n} \frac{1}{l_i} \int_{l_i}^{l_i} (M_i^r(s))^2 ds $$

(5)

the complementary energy computed from the residual forces. Here $l_i, (i = 1, 2, ..., n)$ denotes the length of the beam members, $I_i$ is the moment of inertia of the beam elements $M_i^r(s)$: the residual moment of the beam members, $E$: the Young’s modulus. In case of the moments $M_{i1}^r$ and $M_{i2}^r$ applying at the ends of the beam members the integral function in Equation (5) can be expressed as:

$$ \int_{0}^{l_i} (M_i^r(s))^2 = \frac{1}{3} [(M_1^r)^2 + (M_1^r)(M_2^r) + (M_2^r)^2]. $$

(6)

Applying the Equation (4) the plastic deformations constrained if an appropriate limit value $C_{r0}$ is introduced.

$$ \frac{1}{6E} \sum_{i=1}^{n} \frac{l_i}{l_i} [(M_1^r)^2 + (M_1^r)(M_2^r) + (M_2^r)^2] - C_{r0} \leq 0 $$

(7)

A limit function $G(.)$ can be introduced by the use of Equation (7):

$$ G(C_{r0}, M^r) = C_{r0} - \frac{1}{6E} \sum_{i=1}^{n} \frac{l_i}{l_i} [(M_1^r)^2 + (M_1^r)(M_2^r) + (M_2^r)^2]. $$

(8)

2. Reliability-based residual plastic limit theorem

Introducing the force method and using the basic concepts of the reliability methods the failure of the structure can be defined by $X_R \leq X_S$. Let $X_R$ expresses the non-negative constrain for the statically admissible forces $X_S$ with probability density functions $f_R(x_R)$ and $f_S(x_S)$, respectively.

The probability of failure can be computed from the following equation:
\[ P_f = P[X_R \leq X_S] = \int_{x_R \leq x_S} f_R(x_R) f_S(x_S) dx_R dx_S. \]  

(9)

For most distributions of \( X_R \) and \( X_S \), the above integrals will have to be evaluated numerically. An alternative formulation of the above problem is in terms of the so-called limit state function (LSF) defined by

\[ G(X_R, X_S) = X_R - X_S \leq 0. \]  

(10)

Noting that \( G \leq 0 \) defines the failure event, the probability of failure is given by

\[ P_f = F_G(0) \]  

(11)

where \( F_G(0) \) is the cumulative distribution function of the limit state surface. This formulation offers a convenient solution when both \( X_R \) and \( X_S \) have the normal distribution \( N(\mu_{X_R}, \sigma_{X_R}^2) \) and \( N(\mu_{X_S}, \sigma_{X_S}^2) \) respectively. In that case, \( G \) is also normal with the mean value equal to \( \mu_G = \mu_{X_R} - \mu_{X_R} \) and variance \( \sigma_G^2 = \sigma_{X_R}^2 + \sigma_{X_S}^2 \), hence,

\[ P_f = \Phi \left( -\frac{\mu_G}{\sigma_G} \right) = \Phi(-\beta) \]  

(12)

where \( \Phi(.) \) is the standard normal cumulative density function and \( \beta = \mu_G / \sigma_G \) is reliability index or safety index. The above formula can be graphically interpreted as it is shown in Figure 1. Note that this is an exact solution as long as \( X_R \) and \( X_S \) are normal.

![Figure 1: Graphical interpretation of Equation (12)](image-url)
The failure probability of the structural system is computed by the following integral:

\[
P_f = P[X \in D_f] = P[G(x) \leq 0] = \int_{D_f} f_X(x) dx.
\]  
(13)

Here \(D_f\) is domain of failure. Let assumed that due to the uncertainties the bound for the magnitude of the complementary strain energy of the residual forces is given randomly and for sake of simplicity it follows the normal distribution with given mean value \(\bar{C}_{r0}\) and standard deviation \(\sigma_w\). Due to the number of the probabilistic variables (here only single) the probability of the failure event can be expressed in a closed integral form:

\[
P_{f,\text{calc}} = \int f(\bar{C}_{r0}, \sigma_w) dx.
\]  
(14)

By the use of the strict safety index a reliability condition can be formed:

\[
\beta_{target} - \beta_{calc} \leq 0
\]  
(15)

where \(\beta_{target}\) and \(\beta_{calc}\) are calculated as follows:

\[
\beta_{target} = -\Phi^{-1}(P_{f,\text{target}})
\]  
(16)

and

\[
\beta_{calc} = -\Phi^{-1}(P_{f,\text{calc}}).
\]  
(17)

Here \(\Phi^{-1}\): inverse cumulative distribution function (so called probit function) of the Gaussian distribution. (Due to the simplicity of the present case the integral formulation is not needed, since the probability of failure can be described easily with the distribution function of the normal distribution of the stochastic bound \(\bar{C}_{r0}\)).
3. Elastic-plastic modeling of the pile foundation

Broms (1964) proposed that short and long piles have different failure modes. A short free-head pile rotates or tilts to a point located close to its toe and passive resistance extends above and below the point of rotation (Figure 2(a)). For long free-head pile, the passive resistance is large and pile cannot rotate or tilt. The lower part stays almost vertical and the upper portion deflects in flexure. Failure occurs when the maximum bending moment exceeds the yield strength of the pile section and a plastic hinge forms at the point of maximum bending moments as shown in Figure 2(b).

Consider a constant pile cross section for a free-head long pile, a plastic hinge with a plastic moment of \(\overline{M^p}\) will form at a depth \(z_{max}\) that has no shear force. The \(\overline{M^p}\) can be calculated using elastic-plastic solutions proposed by Guo (2006) for cohesionless soil.

\[
\frac{\overline{M^p}}{A_L} = \frac{1}{n+2} \left[ \alpha_0^{n+1} + (n + 1) \frac{H}{A_L} \right]^{n+2} - \left( \frac{\alpha_0^{n+2}}{n+2} + \frac{\alpha_0 H}{A_L} \right) + \frac{M_e}{A_L} \tag{18}
\]

Where \(A_L = \gamma' N_g d^{1-n}\) associated with the limiting force profile for cohesionless soil; \(\gamma'\) effective density of overburden soil; \(N_g\) gradient.
to correlate soil undrained strength; \( d \) the outer diameter of an cylinder pile; \( n \) power for the limiting force profile; \( \alpha_0 \) equivalent depth to take into the consideration the resistance at the ground surface; \( H \) lateral load and \( M_e = He \), \( e \) the distance from lateral load to the mudline.

4. **Shakedown analysis and design methods**

Considering load parameters \( m_1 \geq 0, m_2 \geq 0 \) in Figure 3 when long pile in cohesionless soil subjected to two separate constant loads \( P_1 \) and \( P_2 \). For each loading combination \( h_i = [m_1 P_1, m_2 P_2] \) a shakedown load parameter \( m_{sh} \) can be calculated. Using these parameters a limit state curves can be created in \( m_1 \)and \( m_2 \) plane for five load combinations as shown in Figure 4.

![Figure 3. Loads on the pile](image-url)

Pile subjected to lateral loads does not fail, if the load parameters \( m_1 \) and \( m_2 \) locates inside or on the limit state curve.
Considering admissible bending moment fields $M_j$ a statically admissible stable shakedown load parameter $m_{sh}$ can be achieved from the condition that even the maximum bending moment does not overstep the fully plastic moment, i.e. $\max |M_j| \leq \overline{M^P}$.

The solution approach based on the static theorem of shakedown analysis, therefore $M^r$ satisfies the equilibrium equation

$$GM^r = 0.$$ (19)

The above equation certifies that during the loading the structure will not undertake unlimited plastic displacements however, doesn’t give us any information concerning permanent displacements which remain in the structure after shakedown. In order to restrict the permanent displacements complementary strain energy of the residual forces is defined as a general measure of the plastic performance of the structures and the residual deformations are constrained by Equation (8).

The elastic internal moment force equation can be formed from Equation (20).

$$M^e = F^{-1}GK^{-1}m_{sh}h_i$$ (20)
Equation (21) defines the yields condition, the $\overline{M_p}$ can be calculated using Equation (18) which proposed by Guo (2006).

$$-\overline{M_p} \leq M^r + \max M^e \leq \overline{M_p}$$ \hfill (21)

For deterministic method, residual plastic deformations of the pile structures are bounded with applying permissible energy value $C_{r0}$:

$$m_{sh} = \max \begin{cases} GM^r = 0 \\ -\overline{M_p} \leq M^r + \max M^e \leq \overline{M_p} \\ M^e = F^{-1}GK^{-1}m_{sh}h_i \end{cases}$$ \hfill (22)

$$\frac{1}{6E} \sum_{i=1}^{n} \frac{l_i}{l_i^2} [ (M_1^r)^2 + (M_1^r)(M_2^r) + (M_2^r)^2 ] - C_{r0} \leq 0$$

This mathematical optimization problem can be executed by the use of nonlinear algorithm.

Considering probabilistic method the bound can be defined with introducing safety index $\beta$:

$$m_{sh} = \max \begin{cases} GM^r = 0 \\ -\overline{M_p} \leq M^r + \max M^e \leq \overline{M_p} \\ M^e = F^{-1}GK^{-1}m_{sh}h_i \end{cases}$$ \hfill (23)

$$\beta_{target} - \beta_{calc} \leq 0$$

4.1 Theses I.

I.a. Shakedown analysis and design method for pile foundations provided in the case of limited residual strain energy capacity.

I.b. I considered an extended shakedown analysis and design method for pile foundations with limited residual strain energy capacity under probabilistic conditions.

5. Plastic limit analysis and design of laterally loaded pile structures

The static principle of limit analysis states that: any statically admissible stable load multiplier \( m_s \) of the elasto-plastic structures is less than or equal to the collapse load multiplier \( m_p \), i.e.:

\[
m_s \leq m_p.
\] (24)

Since the exact value of \( m_p \) is equal to the maximum value of \( m_s \), the problem based on the static principle can be define as follows:

for deterministic problems residual plastic behavior of structures are limited with applying permissible energy value \( C_{r0} \):

\[
\begin{align*}
m_s &= \max \\
GM^p + m_s h_i &= 0 \\
-M^p &\leq \max M^p \leq M^p \\
M^e &= F^{-1}GK^{-1}m_s h_i
\end{align*}
\]

\[
\frac{1}{6E} \sum_{i=1}^{n} \frac{1}{l_i} \left[ (M^r_1)^2 + (M^r_2)(M^r_3) + (M^r_2)^2 \right] - C_{r0} \leq 0
\] (25)

The equations have the same meaning as defined in the previous parts.

In the case of probabilistic problems bound can be defined with introducing reliability index \( \beta \):

\[
\begin{align*}
m_s &= \max \\
GM^p + m_s h_i &= 0 \\
-M^p &\leq \max M^p \leq M^p \\
M^e &= F^{-1}GK^{-1}m_s h_i
\end{align*}
\]

\[
\beta_{target} - \beta_{calc} \leq 0
\] (26)

5.1 Theses II.

II.a. Plastic Limit Analysis and Design Method for pile foundations provided in the case of limited residual strain energy capacity.
II.b. I considered an extended Plastic Limit Analysis and Design Method for pile foundations with limited residual strain energy capacity under probabilistic conditions.


6. Topology optimization in case of uncertain load positions

Recently the topology optimization is very popular topic in the expanding field of optimal design, the majority of the papers deals with deterministic problems or reliability analysis. The reason of introduction stochastic programming theory, more generally, probabilistic notation is to attempt to consider in a more rational way the fact that the precise strength of a structure is not known, among the constraints there are probabilistic inequalities and perhaps even more importantly, that the loadings applied to the structure are not known with any degree of precision. Lógó (2005, 2007) and Lógó et.al (2009) elaborated a rather powerful method for stochastic topology optimization where the magnitude of the loads or the compliance limit are given by their mean values, covariances and distribution functions. Applying an appropriate approximation for the loading uncertainites the stochastic expressions are substituted by equivalent one. This work is a continuation of the above cited papers.

6.1 Approximation of a probabilistic expression

From the literature the following theory is known (Prekopa (1995), Lógó (2007)). If $\xi_1, \xi_2, ..., \xi_n$ have a joint normal distribution, then the set of $\mathbf{x} \in \mathbb{R}^n$ vectors satisfying

$$P(x_1 \xi_1 + x_2 \xi_2 + ... + x_n \xi_n \leq 0) \geq q;$$

(27)

is the same as those satisfying

$$\sum_{i=1}^{n} x_i \mu_i + \Phi^{-1}(q) \sqrt{\mathbf{x}^T K_{ov} \mathbf{x}} \leq 0$$

(28)
where $\mu_i = E(\xi_i)$, $(i = 1,2,...,n)$ is the mean value of the randomly given element $\xi_i$, $K_{ov}$ is the covariance matrix of the random vector $\xi^T = (\xi_1,\xi_2,...,\xi_n)$, $q$ is a fixed probability and $0 < q < 1$, $\Phi^{-1}(q)$ is the inverse cumulative distribution function (so called probit function) of the normal distribution. Expression (28) is convex, the proof can be found in Prekopa (1995).

6.2 Thesis III.

III.a. A new type of probabilistic optimal topology design method investigated where the positions of the loads are given randomly and they can have linear relationship. The calculation of compliance value is based on the assumption that external loads have uncertainties.

Publications related to the thesis: [10]

7. Publications cited in the theses


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8. References


